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Abstract. We study realizations of *polynomial* deformations of the $sl(2, \Re)$-Lie algebra in terms of differential operators strongly related to bosonic operators. We also distinguish their finite- and infinite-dimensional representations. The linear, quadratic and cubic cases are explicitly visited but the method works for arbitrary degrees in the polynomial functions. Multi-boson Hamiltonians are studied in the context of these '*nonlinear*' Lie algebras and some examples dealing with *quantum optics* are pointed out.

1. Introduction

Lie groups and algebras [1] are of great importance in quantum physics [2]. In particular, they are associated with symmetry properties of physical systems and always improve the understanding of many applications. In fact, such developments are mainly based on *linear* Lie algebras, but it has become increasingly evident that there is no physical reason for symmetries to be only linear ones, leading to the observation that Lie theory may therefore be too restrictive. Very recent studies have visited important classes of *nonlinear* symmetries. Among these, let us only mention the nonlinear finite *W*-symmetries [3] described by *nonlinear Lie algebras* that we define as 'generalizations of ordinary Lie algebras containing different order products of the generators on the right-hand side of the defining brackets without violating Jacobi identities' [4]. Another recent approach [5, 6] has considered nonlinear angular momentum theories fundamentally developed from nonlinear extensions of the (real form $sl(2, \Re)$ of the) complex Lie–Cartan algebra A_1. These two contents [3, 5, 6] are not independent [7] and cover an important set of contributions on the corresponding *representations* of nonlinear Lie algebras [4–14] where, more particularly, *quadratic* and *cubic* nonlinearities have been extensively handled.

The main purpose of this paper is to discuss and put forward possible realizations of nonlinear algebras by *differential* operators, such representations also being of primary interest in connection with physical applications described by typical quantum Hamiltonians. As an

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example, we choose to study *polynomial* deformations of the linear $sl(2, \Re)$-algebra written in the form

$$[J_0, J_\pm] = \pm J_\pm \tag{1.1}$$

$$[J_+, J_-] = P(J_0) \tag{1.2}$$

where, as usual, $J_\pm$ are the well known raising $(+)$ and lowering $(-)$ operators (physically important in the angular momentum theory [15]) and J_0 the diagonal one, $P(J_0)$ being a polynomial function of J_0 with a generally finite degree Δ. Particular nonlinear $sl(2, \Re)$-algebras have already characterized physical applications in quantum mechanics [3] but also in Yang–Mills-type gauge theories, inverse scattering, conformal field theories [16, 17], quantum optics [18, 19], etc.

The interest of realizations in terms of linear differential operators is evidently stressed by the fact that position and momentum operators are strongly related to linear combinations of (harmonic) oscillator creation $(a^\dagger)$ and annihilation (a) operators so that the so-called *bosonic* realizations of nonlinear Lie algebras appear as very rich information. Let us here only recall the Heisenberg algebra characterized by the nonzero commutation relation (in a one-dimensional space)

$$\left[\frac{d}{dx}, x \right] = 1 \tag{1.3}$$

equivalent to the oscillator algebra

$$\left[a, a^\dagger \right] = 1 \tag{1.4}$$

with the corresponding number operator given here by the dilation operator

$$D \equiv x \frac{d}{dx}. \tag{1.5}$$

Such bosonic representations have already been studied for a long time in connection with (linear) Lie algebras [20, 21] but we want to tackle nonlinear ones as already mentioned.

The contents are then distributed as follows. In section 2, we propose a differential realization of the three generators $(J_\pm, J_0)$ satisfying the typical commutation relations (1.1) and (1.2) and we discuss three specific polynomial contexts: the (trivial) linear case (section 2.1), the quadratic context (section 2.2) which is a first nontrivial (nonlinear) case with several physical motivations [4, 8, 10–13] and the cubic case (section 2.3), another specific finite W-algebra once again very well visited [5–7, 9, 13, 22, 23] in connection with different physical applications. An important discussion will also take place in that section: we want to distinguish the possible *finite-dimensional* representations of the different algebras. Moreover, we will only be interested in realizations which could be handled in the future, formal ones then being discarded. In section 3, as an application, we visit families of multi-boson Hamiltonians [19] and point out some of them admitting typically such nonlinear Lie algebras as spectrum generating algebras.

Our units are taken with the constant $\hbar$ equal to unity.

2. Differential realizations of the generators

Let us realize the nonlinear algebra (1.1), (1.2) in terms of differential operators depending on a real variable x. Such commutation relations suggest that these operators have the following forms:

$$J_+ = x^N F(D) \qquad J_- = G(D) \frac{d^N}{dx^N} \qquad J_0 = \frac{1}{N}(D + c) \tag{2.1}$$

where c is a constant, D is given by equation (1.5) and $N = 1, 2, 3, \ldots$. Besides the starting property (1.3), let us mention the following identities and notations which are very useful in order to manipulate the above operators:

$$[D, x^N] = N x^N \qquad \left[\frac{\mathrm{d}^N}{\mathrm{d}x^N}, D \right] = N \frac{\mathrm{d}^N}{\mathrm{d}x^N} \tag{2.2}$$

and

$$\frac{\mathrm{d}^N}{\mathrm{d}x^N} x^N = \prod_{j=1}^{N} (D + j) = \frac{(D + N)!}{D!} \tag{2.3}$$

$$x^N \frac{\mathrm{d}^N}{\mathrm{d}x^N} = \prod_{j=0}^{N-1} (D - j) = \frac{D!}{(D - N)!}. \tag{2.4}$$

If the relations (1.1) are easily satisfied, the one given in equation (1.2) asks for a general constraint written as

$$F(D - N)G(D - N)\frac{D!}{(D - N)!} - F(D)G(D)\frac{(D + N)!}{D!} = P\left(\frac{1}{N}(D + c)\right). \tag{2.5}$$

If $P(J_0)$ in equation (1.2) is a polynomial of degree Δ in J_0, we evidently should have $\Delta = \deg_D(FG) + N - 1$.

Here we want to point out that the following transformation

$$\hat{J}_+ = J_+ U \qquad \hat{J}_- = U^{-1} J_- \qquad \hat{J}_0 = J_0 \tag{2.6}$$

defines an automorphism of the realization (2.1) if U is a function of D only. Such an automorphism allows us to simplify the functions $F(D)$ and (or) $G(D)$. In the following, we exploit this freedom without loss of generality. It is thus understood that all the realizations explicitly presented here will be valid up to an automorphism of the form (2.6). So, let us fix our choice on

$$G(D) = 1 \tag{2.7}$$

in the constraint (2.5) and let us apply it to specific cases which are particularly meaningful in connection with physical applications [3, 4, 16, 17]: we want to successively treat the three cases of the (*linear*) algebra $sl(2, \Re)$ and the (*nonlinear*) quadratic and cubic deformations of this structure.

2.1. The linear algebra $sl(2, \Re)$

This case corresponds to the well known example where

$$P(J_0) = 2J_0 = \frac{2}{N}(D + c) \tag{2.8}$$

subtending all the very important consequences of the angular momentum theory [15] and its finite-dimensional representations but dealing with a specific basis other than the one in which we are interested here. Let us also mention a recent study [21] on this case which can be strongly related to ours up to specific periodic conditions imposed in that work [21]. Here the function $F(D)$ is chosen as

$$F(D) = -\frac{D!}{N^2(D + N)!}(D + \lambda_1)(D + \lambda_2) \tag{2.9}$$

where λ_1 and λ_2 are x-independent parameters. The constraint (2.5) with the functions (2.7) and (2.9) leads to

$$\lambda_1 + \lambda_2 = 2c + N$$

so that in terms of a single parameter λ [21] we get

$$\lambda_1 = c + \frac{N}{2} + \lambda \qquad \text{and} \qquad \lambda_2 = c + \frac{N}{2} - \lambda. \tag{2.10}$$

Then the discussion of the function $F(D) \equiv (2.9)$ starts with different N-values but by limiting ourselves to nonsingular realizations of the three generators (2.1).

Here, only the values $N = 1, 2$ are admissible ones in that sense and we immediately get that, for $N = 1$,

$$\lambda = \pm(\tfrac{1}{2} - c) \Rightarrow F(D) = -(D + 2c) \tag{2.11}$$

and, for $N = 2$,

$$\lambda = \pm\tfrac{1}{2} \qquad c = \tfrac{1}{2} \Rightarrow F(D) = -\tfrac{1}{4}. \tag{2.12}$$

Let us now give these explicit realizations in the two cases: we respectively have

$$J_+ = -x\left(x\frac{\mathrm{d}}{\mathrm{d}x} + 2c\right) \qquad J_- = \frac{\mathrm{d}}{\mathrm{d}x} \qquad J_0 = x\frac{\mathrm{d}}{\mathrm{d}x} + c \tag{2.13}$$

and

$$J_+ = -\frac{x^2}{4} \qquad J_- = \frac{\mathrm{d}^2}{\mathrm{d}x^2} \qquad J_0 = \frac{1}{2}\left(x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\right). \tag{2.14}$$

Let us also point out that, if we ask for finite-dimensional representations, we have to consider $P(n)$ (the $(n + 1)$-dimensional vector space of polynomials of degree at most n in the x-variable) and to preserve it under the action of the generator J_+ in particular. This requires, in particular, that

$$J_+x^n = F(n)x^{n+N} = 0 \tag{2.15}$$

and thus that

$$F(n) = F(n - 1) = \cdots = F(n - N + 1) = 0. \tag{2.16}$$

Only the case $N = 1$ with $c = -\frac{n}{2}$ is permitted, the corresponding generators (2.13) being already extensively used in the construction of quasi-exactly solvable equations [24, 25].

2.2. The quadratic algebra

If we require (with arbitrary α_i, $i = 1, 2, 3$ up to $\alpha_3 \neq 0$)

$$P(J_0) = \alpha_1 + \alpha_2 J_0 + \alpha_3 J_0^2 \tag{2.17}$$

we are concerned with an already well-visited nonlinear algebra [8, 13] with specific physical interests [4] if $\alpha_1 = 0$, $\alpha_2 = 2$, $\alpha_3 = 4\alpha$ *or* with a $W_3^{(2)}$-algebra [3] if $\alpha_2 = 0$ and $\alpha_3 > 0$. Let us apply our considerations and search for functions

$$F(D) = -d\frac{D!}{(D + N)!}(D + \lambda_1)(D + \lambda_2)(D + \lambda_3) \tag{2.18}$$

satisfying the condition (2.5) with the entries (2.7) and (2.17). This leads to the information

$$d = \frac{\alpha_3}{3N^3} \qquad \lambda_1 + \lambda_2 + \lambda_3 = \frac{3N\alpha_2}{2\alpha_3} + 3c + \frac{3N}{2}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{N^2}{2} + 3cN + \frac{3N^2\alpha_2}{2\alpha_3} + \frac{3Nc\alpha_2}{\alpha_3} + 3c^2 + 3N^2\frac{\alpha_1}{\alpha_3}. \tag{2.19}$$

Due to the specific form of our functions (2.18), we have only three admissible values $N = 1, 2, 3$. We get for $N = 1$

$$d = \frac{\alpha_3}{3} \qquad \lambda_1 = 1 \qquad \lambda_2 = \frac{3}{2}c + \frac{1}{4} + \frac{3}{4}\frac{\alpha_2}{\alpha_3} + \frac{\epsilon}{2}$$

$$\lambda_3 = \frac{3}{2}c + \frac{1}{4} + \frac{3}{4}\frac{\alpha_2}{\alpha_3} - \frac{\epsilon}{2} \tag{2.20}$$

$$\epsilon = \left(-3c^2 + 3c + \frac{1}{4} + \frac{9}{4}\frac{\alpha_2^2}{\alpha_3^2} - \frac{\alpha_2}{\alpha_3}\left(\frac{3}{2} - 3c\right) - 12\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{2}}$$

so that the corresponding differential realization is

$$J_0 = D + c \qquad J_- = \frac{\mathrm{d}}{\mathrm{d}x} \qquad J_+ = -\frac{\alpha_3}{3}x(D + \lambda_2)(D + \lambda_3). \tag{2.21}$$

It will correspond to finite-dimensional representations of dimension $(n + 1)$ iff $\lambda_2 = -n$ or $\lambda_3 = -n$.

For $N = 2$ we have

$$d = \frac{\alpha_3}{24} \qquad \lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3\left(c + \frac{\alpha_2}{\alpha_3}\right) \tag{2.22}$$

and the realization writes

$$J_0 = \frac{1}{2}(D + c) \qquad J_- = \frac{\mathrm{d}^2}{\mathrm{d}x^2} \qquad J_+ = -\frac{\alpha_3}{24}x^2\left(D + 3c + 3\frac{\alpha_2}{\alpha_3}\right). \tag{2.23}$$

leading to infinite-dimensional representations only.

Finally, for $N = 3$, we have

$$d = \frac{\alpha_3}{81} \qquad \lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3$$

$$\frac{\alpha_2}{\alpha_3} = -\frac{2}{3}c + \frac{1}{3} \qquad \frac{\alpha_1}{\alpha_3} = \frac{1}{9}c^2 - \frac{1}{9}c + \frac{2}{27} \tag{2.24}$$

giving, once again, infinite-dimensional representations through the differential realization

$$J_0 = \frac{1}{3}(D + c) \qquad J_- = \frac{\mathrm{d}^3}{\mathrm{d}x^3} \qquad J_+ = -\frac{\alpha_3}{81}x^3. \tag{2.25}$$

Let us end this section by mentioning that the realizations and representations of the *quadratic* algebra [4] characterized by

$$P(J_0) = 2J_0 + 4\alpha J_0^2 \tag{2.26}$$

are readily obtained by fixing the α_i as already specified.

2.3. The cubic or Higgs algebra

The Higgs algebra [22] has also been intensively exploited up till now, in connection with *either* specific models [13, 18, 22, 23] *or* through technical relations with W-algebras [3, 7] as well as with the study of its irreducible representations [5, 6, 9]. Here it is characterized by the polynomial function

$$P(J_0) = 2J_0 + 8\beta J_0^3 \tag{2.27}$$

$\beta(\neq 0)$ being very often interpreted as a deformation parameter. The functions $F(D)$ corresponding to the previous expressions (2.9) and (2.18) take the form

$$F(D) = -f\,\frac{D!}{(D + N)!}(D + \lambda_1)(D + \lambda_2)(D + \lambda_3)(D + \lambda_4). \tag{2.28}$$

We get the information

$$f = \frac{2\beta}{N^4} \qquad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4c + 2N$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = N^2 + 6cN + 6c^2 + \frac{N^2}{2\beta} \qquad (2.29)$$

$$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 = 2cN^2 + 6c^2N + 4c^3 + \frac{cN^2}{\beta} + \frac{1}{2\beta}N^3$$

and have to discuss the cases $N = 1, 2, 3$ and 4. Here are the results for $N = 1$:

$$\lambda_1 = 1 \qquad \lambda_2 = \frac{1}{2} + c + \frac{1}{2}\sqrt{1 + 4c - 4c^2 - \frac{2}{\beta}}$$

$$\lambda_3 = \frac{1}{2} + c - \frac{1}{2}\sqrt{1 + 4c - 4c^2 - \frac{2}{\beta}} \qquad \lambda_4 = 2c, \qquad (2.30)$$

$N = 2$:

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad (2.31)$$

$$\text{if} \quad c = \frac{1}{2} \qquad \lambda_3 = \frac{3}{2} + \frac{1}{2}\sqrt{7 - \frac{8}{\beta}} \qquad \lambda_4 = \frac{3}{2} - \frac{1}{2}\sqrt{7 - \frac{8}{\beta}} \qquad (2.32)$$

$$\text{or} \quad c = \frac{1}{2} + \frac{1}{2}\sqrt{3 - \frac{4}{\beta}} \qquad \lambda_3 = 1 + \sqrt{3 - \frac{4}{\beta}} \qquad \lambda_4 = 2 + \sqrt{3 - \frac{4}{\beta}} \qquad (2.33)$$

$$\text{or} \quad c = \frac{1}{2} - \frac{1}{2}\sqrt{3 - \frac{4}{\beta}} \qquad \lambda_3 = 1 - \sqrt{3 - \frac{4}{\beta}} \qquad \lambda_4 = 2 - \sqrt{3 - \frac{4}{\beta}}, \qquad (2.34)$$

$N = 3$:

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3 \qquad \lambda_4 = 4c \qquad (2.35)$$

$$\text{if} \quad c = 0 \Rightarrow \lambda_4 = 0 \qquad \beta = \tfrac{9}{4} \qquad (2.36)$$

$$\text{or} \quad c = \tfrac{1}{2} \Rightarrow \lambda_4 = 2 \qquad \beta = \tfrac{9}{7} \qquad (2.37)$$

$$\text{or} \quad c = 1 \Rightarrow \lambda_4 = 4 \qquad \beta = \tfrac{9}{4}, \qquad (2.38)$$

$N = 4$:

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3 \qquad \lambda_4 = 4$$
$$c = \tfrac{1}{2} \qquad \beta = \tfrac{16}{11}. \qquad (2.39)$$

Finite-dimensional representations are only possible for $N = 1$ and 2. If $N = 1$, $\lambda_1 = 1$ and $\lambda_2 = -n$ are fixed but λ_3 and λ_4 can take three different sets of values in correspondence with well definite values of c as follows:

$$\text{if} \quad c = -\frac{n}{2} \qquad \lambda_3 = \frac{1}{2} - \frac{n}{2} + \frac{R_1}{2} \qquad \lambda_4 = \frac{1}{2} - \frac{n}{2} - \frac{R_1}{2} \qquad (2.40)$$

with

$$R_1 = \left(1 - 2n - n^2 - \frac{2}{\beta}\right)^{1/2} \qquad (2.41)$$

$$\text{if} \quad c = -\frac{n}{2} + \frac{1}{2}R_2 \qquad \lambda_3 = -n + R_2 \qquad \lambda_4 = 1 + R_2 \qquad (2.42)$$

with

$$R_2 = \left(-n^2 - 2n - \frac{1}{\beta} \right)^{1/2} \tag{2.43}$$

$$\text{if} \quad c = -\frac{n}{2} - \frac{1}{2}R_2 \qquad \lambda_3 = -n - R_2 \qquad \lambda_4 = 1 - R_2. \tag{2.44}$$

Let us point out that these three families correspond to the three ones obtained in the angular momentum basis [5] associated with $n = 2j$.

If $N = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -n$, $\lambda_4 = -n + 1$ are determined so that we get fixed values for c and β given by

$$c = -\frac{n}{2} \qquad \beta = -\frac{4}{n^2 + 2n - 2} \tag{2.45}$$

leading to supplementary representations recently obtained in [6].

The method developed here can evidently be applied to any degree of nonlinearities in the polynomials characterizing equation (1.2) depending on specific questions and applications we want to consider. In conclusion of this section, let us consider the q-deformation $\mathcal{U}_q(sl(2, \mathfrak{R}))$ of $sl(2, \mathfrak{R})$ which corresponds to the commutation relation (1.2) written on the form

$$[J_+, J_-] = [2J_0] \tag{2.46}$$

where the bracket refers to the usual notation [26]

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad q = e^\gamma \tag{2.47}$$

leading to

$$[2J_0] = \left[\frac{2}{N}(D + c) \right] = \frac{1}{sh\gamma} sh \frac{2\gamma}{N}(D + c) \tag{2.48}$$

where γ characterizes the deformation parameter of this *quantum* algebra. In our differential realization (2.1) with the condition (2.7), the new functions $F(D)$ take the form

$$F(D) = -\frac{1}{2sh^2 \gamma} \frac{D!}{(D + N)!} ch \left(\frac{2\gamma}{N}(D + c) + \gamma \right) \tag{2.49}$$

also rewritten [27] as

$$F(D) = -\frac{1}{2sh^2\gamma} \frac{D!}{(D + N)!} \prod_{k=1}^{\infty} \left(1 + \left(\frac{2\gamma(2D + 2c + N)}{N(2k + 1)\pi} \right)^2 \right). \tag{2.50}$$

No admissible cases can thus be discussed and all the representations are infinite-dimensional (as expected), c still being a free parameter.

3. Some examples from quantum optics

A very recent result [18] concerning a location of fundamental supersymmetry [28] in multi-boson Hamiltonians [19] suggests to learn if our differential approach can be useful through the exploitation of nonlinear Lie algebras seen as spectrum generating algebras [29] of these Hamiltonians as it was the case in quantum optics in particular.

We thus propose to consider the following family of Karassiov–Klimov Hamiltonians [19] given by

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g(a_1^\dagger)^s a_2^r + \bar{g} a_1^s (a_2^\dagger)^r \tag{3.1}$$

where $0 \leqslant r \leqslant s$, ω_1 and ω_2 refer to angular frequencies of two *independent* harmonic oscillators characterized by annihilation (a_1, a_2) and creation $(a_1^\dagger, a_2^\dagger)$ operators verifying the usual Heisenberg relations (1.4).

With the definitions

$$R_0 \equiv \frac{1}{r+s}(ra_1^\dagger a_1 + sa_2^\dagger a_2) = \frac{1}{r+s}(rN_1 + sN_2) \tag{3.2}$$

$$J_0 \equiv \frac{1}{r+s}(a_1^\dagger a_1 - a_2^\dagger a_2) \qquad J_+ \equiv (a_1^\dagger)^s a_2^r \qquad J_- = a_1^s (a_2^\dagger)^r \tag{3.3}$$

the Hamiltonian (3.1) becomes

$$H = (\omega_1 + \omega_2)R_0 + (\omega_1 s - \omega_2 r)J_0 + gJ_+ + \bar{g}J_- \tag{3.4}$$

and we already notice that

$$[R_0, J_0] = [R_0, J_\pm] = 0 \qquad [J_0, J_\pm] = \pm J_\pm \tag{3.5}$$

for arbitrary values of r and s. The Hamiltonian (3.1) suggests an infinite-dimensional basis $\{|n_1, n_2\rangle \; n_1, n_2 = 0, 1, 2, \ldots\}$ for which the eigenvalues of R_0 (noted by j in the following) are of the form $j = (rn_1 + sn_2)/(r+s)$.

3.1. Towards the (cubic) Higgs algebra

In connection with the results [18] enhancing the Higgs algebra as a spectrum generating algebra of the Hamiltonian (3.4), we have to add to equation (3.5) the requirement that the commutator $[J_+, J_-]$ is up to a renormalization of $J_\pm$, of the form (1.2) with (2.27). It was realized in [18] that this is possible only for $r = s = 2$. Here the commutator reads

$$[J_+, J_-] = -64J_0^3 + 8J_0(2R_0^2 + 2R_0 - 1). \tag{3.6}$$

A comparison of (2.27) with (3.6) further indicates that the parameter β should be of the form

$$\beta = -\frac{4}{4j^2 + 4j - 2} \qquad j = 0, \tfrac{1}{2}, 1, \ldots \tag{3.7}$$

which, we note in passing, is very similar to the condition (2.45) obtained in our classification of finite-dimensional representations of the Higgs algebra.

After calculations, one can show that, out of the four cases $N = 1, 2, 3, 4$, only the $N = 1$- and $N = 2$-contexts (see (2.30)–(2.34)) are possible.

We first discuss the three solutions associated with the context *when $N = 2$*:

$$\text{if} \quad c = \tfrac{1}{2} \qquad \lambda_3 = \tfrac{3}{2} + \tfrac{1}{2}(8j^2 + 8j + 3)^{\frac{1}{2}} \qquad \lambda_4 = 3 - \lambda_3 \tag{3.8}$$

we have

$$J_0 = \frac{1}{2}\left(D + \frac{1}{2}\right) \qquad J_- = \frac{d^2}{dx^2} \qquad J_+ = x^2(D + \lambda_3)(D + 3 - \lambda_3) \tag{3.9}$$

$$\text{if} \quad c = j + 1 \qquad \lambda_3 = 2j + 2 \qquad \lambda_4 = 2j + 3 \tag{3.10}$$

we obtain

$$J_0 = \frac{1}{2}(D + j + 1) \qquad J_- = \frac{d^2}{dx^2} \qquad J_+ = x^2(D + 2j + 2)(D + 2j + 3) \tag{3.11}$$

$$\text{if} \quad c = -j \qquad \lambda_3 = -2j \qquad \lambda_4 = -2j + 1 \tag{3.12}$$

we get

$$J_0 = \frac{1}{2}(D - j) \qquad J_- = \frac{d^2}{dx^2} \qquad J_+ = x^2(D - 2j)(D - 2j + 1). \tag{3.13}$$

Only the realization (3.13)—which exactly corresponds to the discussion leading to eqs. (2.45)—is of finite dimension while those given by equations (3.9) and (3.11) are infinite-dimensional ones.

Let us now insert such realizations inside the Hamiltonians (3.4) with the *ideal* conditions $\omega_1 = \omega_2 = \omega, r = s = 2$ and $g = \bar{g}$ in order to determine if the above three contexts are or are not typical ones of the *linear* algebra $sl(2, \Re)$ or of powers of its generators already obtained in equations (2.13) and (2.14).

With the finite-dimensional realization (3.13), we readily find the corresponding Hamiltonian of the form

$$H = 2\omega j + g(1 + x^4)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(1 - 2j)gx^3\frac{\mathrm{d}}{\mathrm{d}x} + 2j(2j - 1)gx^2. \qquad (3.14)$$

The generators $J_\pm$ in (3.13) are nothing else but the *second* power of the corresponding *linear* $sl(2, \Re)$-generators (2.13). The Higgs *and* $sl(2, \Re)$ algebras are simultaneously spectrum generating algebras of this application developed in a $(2j+1)$-dimensional space. This context exactly corresponds to the Karassiov–Klimov model and its deduced supersymmetric features [18]: the double degeneracy of the energy eigenvalues is clearly explained here through the above powers of the linear $sl(2, \Re)$-generators as well as the possible construction of two supercharges [30].

Up to the fact that the realization is infinite dimensional, the results (3.10) and (3.11) lead once again to the conclusion that we are dealing with second powers of the generators of the linear $sl(2, \Re)$-algebra, this context being characterized by the Hamiltonian

$$H = 2\omega j + g(1 + x^4)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (4j + 6)gx^3\frac{\mathrm{d}}{\mathrm{d}x} + (4j^2 + 10j + 6)gx^2. \qquad (3.15)$$

The third case corresponding to equations (3.8) and (3.9) is more interesting. Indeed, we get the Hamiltonian

$$H = 2\omega j + g(1 + x^4)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 4gx^3\frac{\mathrm{d}}{\mathrm{d}x} - \left(2j^2 + 2j - \frac{3}{2}\right)gx^2. \qquad (3.16)$$

Here we notice that our raising operator J_+ cannot be related to the generators of the linear $sl(2, \Re)$ algebra. Such a family of applications is thus characterized by the (cubic) Higgs algebra seen as the spectrum generating algebra of interest, this context being typical once again of infinite-dimensional realizations. Moreover, let us point out that this Hamiltonian (3.16) is a Hermitean operator while the previous expressions (3.14) and (3.15) were not.

The above discussion can evidently be completed by the $N = 1$-context through the values (2.26) with β given once again by equation (3.7). Here also, we have found (finite- or infinite-dimensional) realizations entering a (non-Hermitean) Hamiltonian admitting the Higgs algebra as spectrum generating algebra without any connection with the linear case or its powers. It reads

$$H = 2\omega j + 16gx^4\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 64g(c + 1)x^3\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g\left(1 + 16\left(6c^2 + 6c - \frac{1}{2}j^2 - \frac{1}{2}j + \frac{9}{4}\right)x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}$$
$$+ 32gc(2c^2 - \tfrac{1}{2}j^2 - \tfrac{1}{2}j + \tfrac{1}{4})x \qquad (3.17)$$

and we notice that it includes third orders of derivatives as well as complex energies so that it is not appealing in connection with physical applications.

In conclusion of this section, let us point out that the above discussion gives us some *academic* results ensuring the role of spectrum generating algebras played by the Higgs algebra in the models characterized by the Hamiltonians (3.14)–(3.17) but, through their explicit forms, we immediately see that such Hamiltonians cannot lead us, in the infinite-dimensional cases,

to Schrödinger-like equations which are easy to handle. For such reasons we want to exploit another Hamiltonian (3.4) as given in the following section.

3.2. Towards the quadratic algebra

Instead of $r = s = 2$, let us study the relations (3.1)–(3.5) when $r = 1$, $s = 2$, resulting in the Hamiltonian

$$H = (\omega_1 + \omega_2)R_0 + (2\omega_1 - \omega_2)J_0 + g J_+ + \bar{g} J_- \tag{3.18}$$

$$J_0 = \tfrac{1}{3}(N_1 - N_2) \qquad R_0 = \tfrac{1}{3}(N_1 + 2N_2). \tag{3.19}$$

Here the supplementary commutation relation becomes

$$[J_+, J_-] = P(J_0) = 12J_0^2 - 3R_0(R_0 + 1) \tag{3.20}$$

so that we have to consider the polynomial (2.17) for

$$\alpha_1 = -3j(j + 1) \qquad \alpha_2 = 0 \qquad \alpha_3 = 12. \tag{3.21}$$

Let us stress that this quadratic algebra is thus a $W_3^{(2)}$-algebra [3] of special interest, appearing here, in connection with *quantum optical models*.

After calculations, it appears (as for section 2.2) that the $N = 3$-context is impossible but the others $N = 2$ and $N = 1$ are possible and lead to three different differential realizations. When $N = 2$, if $c = -j$, we get

$$J_0 = \frac{1}{2}(D - j) \qquad J_- = \frac{d^2}{dx^2} \qquad J_+ = -\frac{1}{2}x^2(D - 3j) \tag{3.22}$$

and if $c = j + 1$, we obtain

$$J_0 = \frac{1}{2}(D + j + 1) \qquad J_- = \frac{d^2}{dx^2} \qquad J_+ = -\frac{1}{2}x^2(D + 3j + 3). \tag{3.23}$$

In the case $N = 1$, the parameter c is arbitrary and we get

$$J_0 = D + c \qquad J_- = \frac{d}{dx} \qquad J_+ = -4x\left(D^2 + \left(3c + \frac{1}{2}\right)D + 3c^2 - \frac{3j^2}{4} - \frac{3j}{4}\right). \tag{3.24}$$

Choosing again $\omega_1 = \omega_2 = \omega$ and $g = \bar{g}$, the Hamiltonian (3.18) leads to three specific models in correspondence with the choices (3.22)–(3.24). We respectively get

$$H_1 = \frac{3}{2}\omega j + g\frac{d^2}{dx^2} + \frac{1}{2}(\omega x - gx^3)\frac{d}{dx} + \frac{3}{2}gjx^2 \tag{3.25}$$

$$H_2 = \frac{1}{2}\omega(1 + 5j) + g\frac{d^2}{dx^2} + \frac{1}{2}(\omega x - gx^3)\frac{d}{dx} - \frac{1}{2}g(3j + 3)x^2 \tag{3.26}$$

and

$$H_3 = \omega(c + 2j) - 4gx^3\frac{d^2}{dx^2} + \left(g + \omega x - 4g\left(3c + \frac{3}{2}\right)x^2\right)\frac{d}{dx}$$
$$-4g\left(3c^2 - \frac{3j^2}{4} - \frac{3j}{4}\right)x. \tag{3.27}$$

None of these Hamiltonians is Hermitean but H_1 and H_2 are PT-invariant so that they have real spectra [31]. This eliminates H_3 for evident reasons and we have to enlighten the characteristics of H_1 and H_2 in order to determine *the* Hamiltonian which has to play the interesting role in quantum optics.

Let us *first* establish through conventional quantum mechanical methods that the stationary Schrödinger equation corresponding to H_1 is of the form

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + V_1(x)\right)\psi(x) = Eg^{-1}\psi(x) \tag{3.28}$$

with

$$V_1(x) = \frac{1}{32}x^6 - \frac{\omega}{16g}x^4 + \left(\frac{\omega^2}{32g^2} - \frac{3j}{4} - \frac{3}{8}\right)x^2 - \frac{\omega}{4g}\left(3j - \frac{1}{2}\right) \tag{3.29}$$

while the one corresponding to H_2 is characterized by the potential

$$V_2(x) = \frac{1}{32}x^6 - \frac{\omega}{16g}x^4 + \left(\frac{\omega^2}{32g^2} + \frac{3j}{4} + \frac{3}{8}\right)x^2 - \frac{\omega}{4g}\left(5j + \frac{1}{2}\right). \tag{3.30}$$

We thus recover two particular forms of the famous potential of degree six, typical of quasi-exactly solvable equations [24, 25, 32] now appearing in quantum optics. Due to these results we are able to calculate a certain number of eigenvalues of the spectrum as well as their eigenfunctions.

More precisely, by applying the techniques of [25, 32] to the sextic oscillators above, we notice that only the potential $V_1(x)$ gives a quasi-exactly solvable Schrödinger equation leading to the existence of $[\frac{3j}{2}]+1$ exact solutions (where $[y]$ refers to the entire part of y) and corresponding to real energy eigenvalues whose first ones are

$$j = 0 \qquad E_0 = 0 \qquad j = \tfrac{1}{3} \qquad E_1 = \omega \qquad j = \tfrac{2}{3} \qquad E_2 = \tfrac{3}{2}\omega \pm \tfrac{1}{2}(\omega^2 + 8g^2)^{\frac{1}{2}}$$

$$j = 1 \qquad E_3 = \tfrac{5}{2}\omega \pm \tfrac{1}{2}(\omega^2 + 24g^2)^{\frac{1}{2}} \quad \text{etc} \ldots . \tag{3.31}$$

This leads to the conclusion that, with the differential realizations of the generators of the *quadratic* Lie algebra (3.20), the Schrödinger equation (3.28) with (3.29) is the only one associated with a realistic quantum optical model. As a *second* point, we would also like to comment about the supersymmetric properties of the Hamiltonian obtained above. For this purpose, let us consider (3.25) for $j = 0$ and, thus, for the eigenvalue $E_0 = 0$, it simplifies to

$$H_1 = \frac{d^2}{dx^2} - \frac{1}{16}x^6 + \frac{\omega}{8g}x^4 + \left(\frac{3}{4} - \frac{\omega^2}{16g^2}\right)x^2 - \frac{\omega}{4g}. \tag{3.32}$$

Interestingly, this operator can be expressed in terms of a superpotential [28] $W_1(x)$

$$H_+ = H_1 = \frac{d^2}{dx^2} - W_1^2 - \frac{dW_1}{dx} \tag{3.33}$$

with

$$W_1(x) = -\frac{1}{4}x^3 + \frac{\omega}{4g}x. \tag{3.34}$$

The superpartner H_- of (3.33) is given [28] by

$$H_- = \frac{d^2}{dx^2} - W_1^2 + \frac{dW_1}{dx} \tag{3.35}$$

and surprisingly appears to be related to H_2:

$$H_- = H_2|_{n=0}. \tag{3.36}$$

It shows that its nonquasi-exactly solvable context is strongly related to the *missing* fundamental energy $E_0 = 0$ and that it does not correspond to the expected results of the quantum optical model. Let us also point out that these superpartners indicate that (only) for $n = 0$, we describe here an *exact* supersymmetry [28].

A final *third* property that we want to stress is related to the realization (2.13) of the linear $sl(2, \Re)$: we have also noticed that the interesting Hamiltonian (3.25) is once again a simple function of the three generators given in equation (2.13) but expressed in terms of a new variable $z = x^2$. We thus conclude that the linear $sl(2, \Re)$ as well as quadratic algebras we are considering (this latest way being, by far, more straightforward) are simultaneously spectrum generating algebras for the model under study. With $c = -\frac{3j}{4}$ and the realization (2.13) given by

$$j_+ = -z^2 \frac{\mathrm{d}}{\mathrm{d}z} + \frac{3j}{2} z \qquad j_- = \frac{\mathrm{d}}{\mathrm{d}z} \qquad j_0 = z \frac{\mathrm{d}}{\mathrm{d}z} - \frac{3j}{4} \tag{3.37}$$

we find

$$H_1 = \tfrac{9}{4}\omega j + \omega j_0 + g j_+ + (3j + 2) g j_- + 4 g j_0 j_-. \tag{3.38}$$

The above study shows the real importance of those nonlinear structures we have put in evidence in connection, here, with quantum optical models. Let us finally notice that another application of nonlinear algebras has recently appeared [33] in the context of the Calogero–Sutherland model [34].

Acknowledgments

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